CALCULATING THE PLANE WAKE BEHIND A BODY

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The empirical theory of turbulence serves as a basis in this study of velocity, enthalpy, turbulence energy, and mean-squared enthalpy (density) fluctuation profiles along the wake axis. The problem is reduced to a system of ordinary differential equations. The asymptotic behavior of the solution to this system is analyzed. The results of calculations are compared with known test data.

1. It is assumed that the velocity, the enthalpy, and the density in the far wake behind a flat body can be determined from the system of boundary-layer equations

$$
\begin{gathered}
\frac{\partial \rho v_x}{\partial x}+\frac{\partial \rho v_y}{\partial y}=0, \\
\rho v_x \frac{\partial v_x}{\partial x}+\rho v_y \frac{\partial v_x}{\partial y}=\frac{\partial}{\partial y}\left[(\mu+\mu_t) \frac{\partial v_x}{\partial y}\right], \\
\rho v_x \frac{\partial H}{\partial x}+\rho v_y \frac{\partial H}{\partial y}=\frac{\partial}{\partial y}\left\{\left(\frac{\mu}{\mathrm{Pr}}+\frac{\mu_t}{\mathrm{Pr}_t}\right) \frac{\partial H}{\partial y}\right. \\
\left.+\left[\mu\left(1-\frac{1}{\mathrm{Pr}}\right)+\mu_t\left(1-\frac{1}{\mathrm{Pr}_t}\right)\right] v_x \frac{\partial v_x}{\partial y}\right\}, \\
\rho=h_\infty / h \equiv 1 / \tilde{h}, \quad H \equiv h+v_x^2 / 2.
\end{gathered} \tag{1}
$$

All quantities here and in the subsequent analysis are dimensionless.

The balance equation pertaining to mean-square enthalpy fluctuations can be derived conventionally [1] from the energy equation. The latter is

$$
\begin{gathered}
\frac{\partial}{\partial x_j}\left(\overline{\rho h'^2}/2\, v_j+\overline{\rho v_j' h'^2/2}-\overline{\omega_j' h'}\right)=-\overline{\rho v_j' h'} \frac{\partial h}{\partial x_j}-\overline{\omega_j' \frac{\partial h'}{\partial x_j}} \\
-v_j \overline{\rho h'} \frac{\partial h}{\partial x_j}+\overline{\tau_{ij} \frac{\partial v_i}{\partial x_j} h'}+\overline{v_j \frac{\partial p}{\partial x_j} h'}, \quad \omega_j \equiv \frac{\lambda}{c_p} \cdot \frac{\partial h}{\partial x_j}
\end{gathered} \tag{2}
$$

where the bar above a symbol implies averaging and appears only with the second and the third moment.

The last term on the left-hand side $\overline{\omega_j' h'}$ has a value only near solid surfaces [2] and is, therefore, disregarded here. The last term on the right-hand side can be rewritten as

$$
\overline{v_j \frac{\partial p}{\partial x_j} h'}=\overline{v_j' h'} \frac{\partial p}{\partial x_j}+v_j \overline{\frac{\partial p'}{\partial x_j} h'}+\overline{v_j' \frac{\partial p'}{\partial x_j} h'}.
$$

Since the pressure fluctuations are relatively small [1] and there is no pressure gradient within the region studied here, this term will also be disregarded. The third and the fourth term in expression (2) are, according to the estimates at the end of this article, small and negligible in our case. With boundary-layer estimates applied to Eq. (2) and assuming that

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$$\frac{1}{2}\overline{\rho v_y' h'^2} = -\frac{\mu_t}{\mathrm{Pr}_t}\cdot\frac{\partial\theta}{\partial y}, \quad \overline{\omega_j' \frac{\partial h'}{\partial x_j}} = c_2\frac{\mu_t}{\mathrm{Pr}_t}\cdot\frac{\theta}{\delta^2},$$
$$\overline{\rho v_y' h'} = -\frac{\mu_t}{\mathrm{Pr}_t}\cdot\frac{\partial h}{\partial y}, \quad \theta = \overline{h'^2}/2 \approx \overline{\rho h'^2}/2\rho, \tag{3}$$

we arrive at the equation

$$\rho v_x\frac{\partial\theta}{\partial x} + \rho v_y\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu_t}{\mathrm{Pr}_t}\cdot\frac{\partial\theta}{\partial y}\right) + \frac{\mu_t}{\mathrm{Pr}_t}\left(\frac{\partial h}{\partial y}\right)^2 - c_2\frac{\mu_t}{\mathrm{Pr}_t}\cdot\frac{\theta}{\delta^2}. \tag{4}$$

A semiempirical closed equation of kinetic turbulence energy E per unit fluid mass can be obtained in an analogous manner and in the small form as (4). Henceforth we will simplify the notation by writing the equations for E and θ as

$$\rho v_x\frac{\partial S_k}{\partial x} + \rho v_y\frac{\partial S_k}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu_t}{\mathrm{Pr}_t^{k-1}}\cdot\frac{\partial S_k}{\partial y}\right) + \frac{\mu_t}{\mathrm{Pr}_t^{k-1}}\left(\frac{\partial w_k}{\partial y}\right)^2 - c_k\frac{\mu_t}{\mathrm{Pr}_t^{k-1}}\cdot\frac{S_k}{\delta^2}, \quad k=1,2, \tag{5}$$

with

$$S_1 \equiv E, \quad w_1 \equiv v_x, \quad S_2 \equiv \theta, \quad w_2 \equiv h.$$

The boundary conditions for system (1), (5) are

$$y=0, \quad v_y = \frac{\partial v_x}{\partial y} = \frac{\partial h}{\partial y} = \frac{\partial S_k}{\partial y} = 0,$$
$$y\to\infty, \quad v_x\to 1, \quad h\to h_\infty, \quad S_k\to 0, \tag{6}$$
$$x=0, \quad v_x = v_{\text{н}}(y), \quad h = h_{\text{н}}(y), \quad S_k = S_{k\text{н}}(y), \quad k=1, 2.$$

2. An approximate solution to problem (1), (5), (6) is obtained by the method of integral relations. We let

$$\frac{1-v_x}{1-v_0} = \exp\left[-(\eta/\Delta)^2\ln 2\right] \equiv r, \tag{7}$$
$$H_\infty - H = a_1(1-v_x) + a_2(1-v_x^2), \tag{8}$$
$$S_k/S_{k0} = r\left[1 + b_k(\eta/\Delta)^2\right], \quad b_k = \text{const}, \quad k=1, 2, \tag{9}$$

where

$$\eta = \int_0^y \rho dy, \quad \Delta = \int_0^\delta \rho dy. \tag{10}$$

It follows from (8) that

$$1/\rho = \tilde{h} = 1 - \alpha_1 u_0 r + \alpha_2 u_0^2 r^2,$$
$$\alpha_1 \equiv (\varkappa - 1)M_\infty^2(a_1 + 2a_2 - 1), \quad \alpha_2 \equiv (\varkappa-1)M_\infty^2(a_2 - 1/2). \tag{11}$$

We note that representations (7)-(9) satisfy all constraints imposed on the corresponding functions at y = 0 and $y\to\infty$ in accordance with (6). We will determine the unknown functions of the longitudinal coordinate Δ: v_0, a_1, a_2, S_{k0} when the integral relations found from the equations of motion and energy as well as from Eq. (5) are satisfied and when the equations of motion and energy are exactly satisfied on the wake axis.

On the basis of (7)-(9) and under the assumption that μ_t is a function of the longitudinal coordinate only, the integral relations will be written as

$$\Delta = \sqrt{\frac{\ln 2}{4\pi}}\cdot\frac{c_x}{u_0(1-u_0/\sqrt{2})}, \quad c_x \equiv 4\int_0^\infty \rho v_x(1-v_x)\,dy, \tag{12}$$

$$a_1 + a_2 f(u_0) = c_t/c_x, \quad c_t \equiv 4\int_0^\infty \rho v_x(H_\infty - H)\,dy, \tag{13}$$

$$\frac{dS_{k0}}{dx}=-S_{k0}\left\{\frac{d\ln g_k}{dx}+c_k\frac{\mu_t}{\Delta g_k \mathrm{Pr}_t^{k-1}}\left[1-\frac{\alpha_1 u_0}{\sqrt{2}}+\frac{\alpha_2 u_0^2}{\sqrt{3}}\right.\right.$$
$$\left.\left.+\frac{b_k}{\ln 2}\left(1-\frac{\alpha_1 u_0}{2\sqrt{2}}+\frac{\alpha_2 u_0^2}{3\sqrt{3}}\right)\right]\right\}+\frac{4\ln 2}{\sqrt{\pi}\left[2\mathrm{Pr}_t(\varkappa-1)\mathrm{M}_\infty^2\right]^{k-1}}\cdot\frac{\mu_t u_0^2}{\Delta g_k}I_k, \tag{14}$$

where

$$u_0\equiv 1-v_0,\quad f(u_0)\equiv\frac{2-3u_0/\sqrt{2}+u_0^2/\sqrt{3}}{1-u_0/\sqrt{2}},$$
$$g_k\equiv\Delta\left[1+\frac{b_k}{2\ln 2}-\frac{u_0}{\sqrt{2}}\left(1+\frac{b_k}{4\ln 2}\right)\right], \tag{15}$$
$$I_k\equiv\int_0^1 \rho r\sqrt{-\ln r}\,(\alpha_1-2\alpha_2 u_0 r)^{2(k-1)}\,dr,$$

and $\rho(r)$ is given by (11).

Letting y = 0 in the equations of motion and energy, we obtain with the aid of (7)-(9), (12)-(13):

$$\frac{du_0}{dx}=-2\ln 2(\mu_0+\mu_t)\frac{\rho_0 u_0}{\Delta^2(1-u_0)}, \tag{16}$$

$$\frac{da_2}{dx}=\frac{du_0/dx}{2u_0-u_0^2-u_0 f}\left\{a_2 u_0\frac{df}{du_0}-a_1-2a_2(1-u_0)+\frac{1}{\mu_0+\mu_t}\left[\left(\frac{\mu_0}{\mathrm{Pr}}+\frac{\mu_t}{\mathrm{Pr}_t}\right)(a_1+2a_2 v_0)\right.\right.$$
$$\left.\left.+\left(\mu_0\left(1-\frac{1}{\mathrm{Pr}}\right)+\mu_t\left(1-\frac{1}{\mathrm{Pr}_t}\right)\right)v_0\right]\right\}. \tag{17}$$

The system of three differential equations (14), (16), (17) yields five unknown functions. The initial conditions for this system are defined by an approximation of functions $v_i(y)$, $h_i(y)$, $S_{ki}(y)$ given at the first section and by relations (7)-(9).

If $\mathrm{Pr}=\mathrm{Pr}_t=1$, then system (1) has the particular integral

$$H=C_1+C_2 v_x,\quad C_1=\mathrm{const},\quad C_2=\mathrm{const}. \tag{18}$$

If $v_i(y)$ and $h_i(y)$ satisfy this relation, then the latter is the solution to the problem. It is not difficult to see that, under the given conditions, the integral method will also yield relation (18). Indeed, if $\mathrm{Pr}=\mathrm{Pr}_t=1$, then (17) becomes

$$\frac{da_2}{du_0}-a_2\frac{u_0 df/du_0}{2u_0-u_0^2-u_0 f}=0. \tag{19}$$

By virtue of (18), the second of said conditions yields $a_{2i}=0$. The solution to the linear homogeneous equation (19) with such an initial condition is $a_2\equiv 0$. It becomes evident from (8), then, that (18) is satisfied throughout the wake region.

We will next examine the asymptotic behavior of the sought functions as $u_0\to 0$, considering for simplicity that

$$\mu\ll\mu_t,\quad \mu/\mathrm{Pr}\ll\mu_t/\mathrm{Pr}_t.$$

Changing in (17) to the independent variable u_0 and using (12), we find at $u_0\to 0$

$$a_2\sim\left(\frac{c_t}{c_x}-1\right)q/u_0,\quad q\equiv\frac{(1-\mathrm{Pr}_t)\sqrt{2}}{2\sqrt{2}-1+\mathrm{Pr}_t(1-\sqrt{2})}$$

(the symbol ~ denotes asymptotic equality). Taking this relation into account, as well as (11) and (13), we find

$$\tilde{h}_0-1\sim(\varkappa-1)\mathrm{M}_\infty^2(c_t/c_x-1)(2q-1)u_0. \tag{20}$$

The behavior of integral (15) at $u_0\to 0$ is described by the expression

$$I_k\sim\sqrt{\frac{\pi\ln 2}{8}}\left[\left(\frac{c_t}{c_x}-1\right)^2 F(\mathrm{Pr}_t)\right]^{k-1},$$

$$F(\mathrm{Pr}_t) \equiv (1-q)^2 - \frac{8}{3}\sqrt{\frac{2}{3}}\,q(1-q) + \sqrt{2}\,q^2. \tag{21}$$

From Eq. (14), with the aid of (12), (16), and (21), we obtain the following expression for the asymptotic value of S_{k0}:

$$\frac{S_{k0}}{\{[(\varkappa-1)\mathrm{M}_\infty^2]^{k-1} w_k - 1\}^2} = \frac{\sqrt{2}(\ln 2)^2}{(2\ln 2 + b_k)(c_k - 2\,\mathrm{Pr}_t^{k-1}\ln 2)}\left[\frac{F(\mathrm{Pr}_t)}{(\varkappa-1)^2\mathrm{M}_\infty^4(2q-1)^2}\right]^{k-1} \equiv \frac{1}{2\gamma_k}. \tag{22}$$

The asymptotic behavior of $u_0(x)$ is found from (16) with the aid of (12). The coefficient of turbulent viscosity will be determined from the formula

$$\mu_t = k\rho_0 u_0 \delta. \tag{23}$$

After necessary calculations, we obtain

$$u_0 \sim \sqrt{\frac{c_x}{8k\sqrt{\pi\ln 2\,x}}}. \tag{24}$$

This formula agrees exactly with results of the asymptotic analysis made by Townsend [2] (in the Townsend notation $k = 1/R_T\sqrt{2\ln 2}$) for an incompressible fluid.

Relation (24) yields the value of the empirical constant k, with the aid of the test data on the u_0-distribution at far distances from the body. According to [2] and [3], k = 0.065.

3. The system of equations (16), (17), (14) was integrated numerically on a digital computer by the Runge–Kutta method. The starting values and the values at infinity were taken from the test in [4]. The coefficient of molecular viscosity was calculated by the Sutherland formula with Pr = 0.75 and $\mathrm{Pr}_t = 0.7$.

Distributions of the velocity defect u_0 (curve 1), of the enthalpy $(\bar{h}_0 - 1)/10$ (curve 2), and of the wake half-width δd (expressed in du s, as in [4], curve 3) are shown and compared with test data in Fig. 1. Up to the section $x_{tr} = 550$ calculations did not include turbulent viscosity. The calculated values corresponding to μ_t according to (23) are indicated by solid lines. This method of calculating the transition region leads to a discontinuity in the total viscosity coefficient at $x = x_{tr}$ (which corresponds to a break in the solid curves in Fig. 1).

In order to avoid such a discontinuity, an attempt was made to calculate the coefficient of turbulent viscosity by the formula

$$\mu_t = k\rho_0\sqrt{2\gamma_1 E_0}\,\delta$$

(We note that, as a consequence of (22), this formula leads to the same asymptotic relation (24) for the velocity defect as formula (23).) In this case the total viscosity is not discontinuous at point $x = x_{tr}$, because the turbulent viscosity rises (together with the turbulence energy) above zero. The corresponding calculated values are shown in Fig. 1 by dashed lines. These lines, in a strict sense, blend smoothly with the curves representing the laminar region, while calculations yield a rather sharp transition to the curves representing the turbulent region. Such a trend of these curves is caused by the fast initial increase in the ratio of turbulence energy to velocity defect, as shown in Fig. 2. The value of constant c_1 is taken from [5], where the autonomous profile of turbulence energy has been analyzed and where the test value $2\gamma_1 \approx 8.1$ [2] has been found to correspond to $c_1 = 3\cdot 2\ln 2 \approx 4.15$. With γ_1 and c_1 known, the value of constant b_1 is found from formula (22) as $b_1 \approx 0.6$.

The mean-squared enthalpy fluctuations have been calculated by Eq. (14) with k = 2. The values of c_2 and b_2 necessary for the calculation are determined as follows. We have $2\gamma_2 = 4.1$, according to the tests in [4]. An analysis of the autonomous problem analogous to the problem in [5], which concerns the profile mean-squared enthalpy fluctuations in a wake, shows that $c_2 = 2.1$ corresponds to such a value of $2\gamma_2$. As a consequence of (22), we have $b_2 = 1.1$.

The mean-squared density fluctuations are, on the basis of the hypothesis in [1]

$$\rho'/\rho \approx -h'/h \tag{25}$$

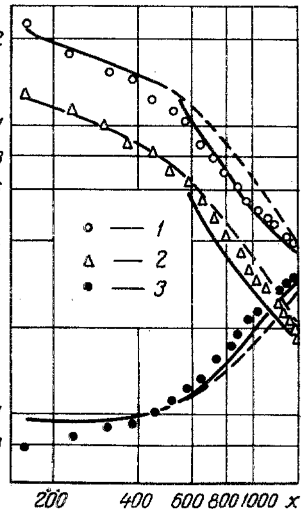

Fig. 1. Basic wake parameters: u_0 (1), $(\bar{h}_0 - 1)/10$ (2), δd (3).

expressed in terms of θ as follows:

$$\sqrt{\rho'^2} = (\varkappa - 1)\, M_\infty^2 \rho \sqrt{2\theta}\,.$$

A mean-squared density fluctuation profile along the wake axis is shown in Fig. 3, referred to the density defect. On the same diagram we also show test data. Evidently, the measured and the calculated increase of density fluctuations are in satisfactory agreement.

We will now estimate the values of the third and the fourth term on the right-hand side of Eq. (2). On the basis of hypothesis (25), the third term can be written as

$$\overline{v_j \rho h'}\, \frac{\partial h}{\partial x_j} = -\frac{2\theta}{h}\left(\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y}\right).$$

The ratio of this term to the dissipative term, on the wake axis, will be considered in conjunction with the second of the semiempirical relations (3) and equality (16) (at $\mu_c \gg \mu$):

$$\left|\frac{\overline{v_j \rho h'}\,\partial h/\partial x_j}{\overline{\omega_j' \partial h'/\partial x_j}}\right|_{y=0} = \frac{(2/h_0)\, \rho_0 v_0 dv_0/dx}{c_2 \mu_t / \mathrm{Pr}_t \delta^2}$$

$$= \frac{4\,\mathrm{Pr}_t \ln 2}{c_2} (\varkappa - 1) \frac{M_0^2}{v_0^2} \left(\frac{\rho_0 \delta}{\Delta}\right)^2 u_0. \tag{26}$$

Here $4\mathrm{Pr}_t \ln 2/c_2$ is a quantity close to unity, and $\rho_0\delta/\Delta < 1$ (see (1)). Evidently, this ratio decreases along the longitudinal coordinate. In the calculation described here this ratio decreases from 0.3 at the beginning of the turbulent region down to 0.1. Consequently, at a high Mach number this term of the equation may become large (only within a bounded region).

The fourth term on the right-hand side of Eq. (2) will be estimated as

$$\left|\overline{\tau_{ij} \frac{\partial v_i}{\partial x_j} h'}\right| = \left|\tau_{ij} \overline{\frac{\partial v_i'}{\partial x_j} h'} + \frac{\partial v_i}{\partial x_j} \overline{\tau_{ij}' h'} + \overline{\tau_{ij}' \frac{\partial v_i'}{\partial x_j} \cdot h'}\right|$$

$$\approx \left|\overline{\tau_{ij}' \frac{\partial v_i'}{\partial x_j} h'}\right| \leqslant \sqrt{\overline{\left(\tau_{ij}' \frac{\partial v_i'}{\partial x_j}\right)^2}\, \overline{h'^2}} \approx \rho\varepsilon \sqrt{2\theta}\,, \tag{27}$$

with ε denoting the per unit dissipation of kinetic turbulence energy. This quantity is described by a semiempirical expression analogous to the second one in (3):

$$\rho\varepsilon = c_1 \mu_t E/\delta^2. \tag{28}$$

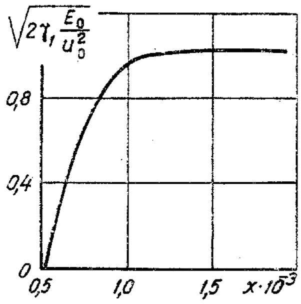

Fig. 2

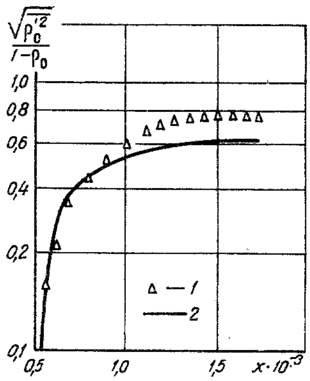

Fig. 3

Fig. 2. Increase in turbulence energy.

Fig. 3. Mean-squared density fluctuations: test in [4] (1), calculation (2).

With (27) and (28) we can write

$$\left|\frac{\overline{\tau_{ij}h'\partial v_i/\partial x_j}}{\overline{\omega_j'\partial h'/\partial x_j}}\right| \approx \frac{c_1 \mathrm{Pr}_t \sqrt{2}}{c_2} \cdot \frac{E_0}{u_0^2} \cdot \frac{\tilde{h}_0 - 1}{\sqrt{\theta_0}} \cdot \frac{u_0^2}{\tilde{h}_0 - 1}.$$

From the last expression, taking into account (20) and (22), we find that the said ratio tends toward zero along the longitudinal coordinate. Under conditions of the problem analyzed here, this ratio does not exceed 3%.

NOTATION

x, y	are the longitudinal and transverse coordinates (referred to the characteristic body thickness, d);
v_x, v_y	are the components of the velocity vector, along the coordinate axes (referred to v_∞);
ρ	is the density (referred to ρ_∞);
h, E	are the enthalpy and kinetic energy of fluctuating motion, per unit fluid mass (referred to v_∞^2);
$\theta \equiv \overline{h'^2}/2$;	
μ, μ_t	are the coefficient of laminar and of turbulent viscosity respectively (referred to $\rho_\infty v_\infty d$);
Pr, Pr_t	are the Prandtl number and Prandtl number for turbulent flow;
p	is the pressure (referred to $\rho_\infty v_\infty^2$);
ε	is the per unit dissipation of kinetic turbulence energy (referred to v_∞^3/d);
δ	is the wake half-width (referred to d);
c_1, c_2, k	are the empirical constants;
γ_k	see Eq. (22);
b_k	are the constants in the S_k-profiles;
a_1, a_2	are the parameters of enthalpy profile;
α_1, α_2	see Eq. (11);
η	is the Dorodnitsyn variable;
Δ	is the transformed wake half-width;
u	is the velocity defect;
Ma	is the Mach number.

Subscripts and Superscripts

∞	denotes an unperturbed stream;
0	denotes the symmetry axis of a wake;
i	denotes the first section;
tr	denotes the transition point;
t	denotes turbulence;
'	denotes fluctuation;
–	denotes an averaged quantity (omitted in symbols denoting the first moments).

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